Theoretical derivation of $1/f$ noise in quantum chaos

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It was recently conjectured that $1/f$ noise is a fundamental characteristic of spectral fluctuations in chaotic quantum systems. This conjecture is based on the behavior of the power spectrum of the excitation energy fluctuations, which is different for chaotic and integrable systems. Using random matrix theory we derive theoretical expressions that explain the power spectrum behavior at all frequencies. These expressions reproduce to a good approximation the power laws of type $1/f$ ($1/f^2$) characteristics of chaotic (integrable) systems, observed in almost the whole frequency domain. Although we use random matrix theory to derive these results, they are also valid for semiclassical systems.

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The study of energy level fluctuations is a basic tool in understanding quantum chaos. The pioneering work of Berry and Tabor showed that the spectral fluctuations of a quantum system whose classical analogue is fully integrable are well described by Poisson statistics [1]. On the other hand, Bohigas *et al.* conjectured that the fluctuation properties of quantum systems which in the classical limit are fully chaotic coincide with those of Random Matrix Theory (RMT) [2]. Recently, a different approach to quantum chaos has been proposed [3]. Considering the sequence of energy levels as a discrete time series, it has been conjectured that chaotic quantum systems are characterized by $1/f$ noise, whereas integrable quantum systems exhibit $1/f^2$ noise. This conjecture is supported by numerical experiments which involve classical Random Matrix Ensembles (RME) and atomic nuclei [3].

In this Letter we present a theoretical derivation of $1/f$ and $1/f^2$ noises in chaotic and regular systems, respectively. We use RMT to derive these results, but they are also valid for semiclassical systems. We present the main steps of the derivation and compare the theoretical results with numerical calculation for RME, an atomic nucleus and a quantum billiard, finding excellent agreement.

For any quantum system the accumulated level density $N(E)$ can be separated into a smooth part $\overline{N}(E)$ and a fluctuating part $\widetilde{N}(E)$. To remove the main trend defined by the former, the energy levels E_i are mapped onto new dimensionless levels $\epsilon_i = \overline{N}(E_i)$. This transformation, called unfolding, allows to compare the statistical properties of different systems or different parts of the same spectrum.

The analogy between the energy spectrum and a discrete time series is established in terms of the δ_q statistic, defined as

$$\delta_q = \sum_{i=1}^{q} (s_i - \langle s \rangle) = \epsilon_{q+1} - \epsilon_1 - q, \qquad (1)$$

where ϵ_i is the i-th unfolded level and $s_i = \epsilon_{i+1} - \epsilon_i$. Once the unfolding is performed, the average nearest level spacing is $\langle s \rangle = 1$. Note that δ_q represents the deviation of the excitation energy of the $(q+1)$-th unfolded level from its mean value. Moreover, it is closely related to the level density fluctuations. Indeed, we can write

$$\delta_q = -\tilde{N}(E_{q+1}), \qquad (2)$$

if we appropriately shift the ground state energy; thus, it represents the accumulated level density fluctuations at $E = E_{q+1}$. Its power spectrum, defined as the square of the modulus of the Fourier transform, shows neat power laws both for fully chaotic and integrable systems, i.e.,

$$P_k^{\delta} \propto \frac{1}{k^{\alpha}}, \qquad (3)$$

but level correlations change the exponent from $\alpha = 2$ for uncorrelated spectra to $\alpha = 1$ for chaotic quantum systems [3].

Notation.– We consider an interval of length $L \gg 1$ containing $N \simeq L$ unfolded energy levels. The fluctuating parts of the level and accumulated level densities are denoted $\widetilde{\rho}(\epsilon)$ and $\widetilde{n}(\epsilon)$, respectively. In addition to the δ_q statistic, we also introduce another discrete function $\widetilde{n}_q = \widetilde{n}(q)$, obtained by sampling the continuous function for integer values of the energy. The Fourier transforms and power spectra of these functions are defined in the usual way [4]. The differences between continuous and discrete functions, and specially between their Fourier transforms, play a relevant role in the following. The notation we use for all these functions, Fourier transforms and power spectra is summarized in Table I.

TABLE I: Summary of the functions, Fourier transforms and power spectra used in this letter

Domain	$\mathbb{R}$	$\mathbb{Z}$	
Function	$\widetilde{n}(\epsilon)$	$\widetilde{n}_q$	δ_q
Fourier transform	$\widehat{n}(\tau)$	$\widehat{n}_k$	$\widehat{\delta}_k$
Power spectrum	$P^n(\tau)$	P^n_k	P^δ_k

Spectral and ensemble averages will be denoted by $\langle\cdot\rangle$, or $\langle\cdot\rangle_\beta$ to distinguish ensemble averages performed in different RME. Here, β stands for the repulsion parameter characterizing the ensemble. In this work we shall consider two RME: the Gaussian Orthogonal Ensemble (GOE) and the Gaussian Unitary Ensemble (GUE) [6]. We have $\beta = 1$ for GOE and $\beta = 2$ for GUE.

Spectral fluctuations.– The main object of this Letter is to obtain explicit expressions of the average value of P^δ_k for chaotic and integrable systems. Except for integrable systems, one of the main features of quantum spectra is that successive level spacings are not independent random variables, but correlated quantities. This property makes exceedingly difficult to work directly with the discrete δ_q sequence. To circumvent this difficulty we profit from the relationship (2). The statistical properties of $\widetilde{n}(\epsilon)$ are usually measured in terms of the spectral form factor, defined as

$$K(\tau) = \left\langle \left| \int d\epsilon \widetilde{\rho}(\epsilon) e^{-i2\pi\epsilon\tau} \right|^2 \right\rangle, \tag{4}$$

that is, as the power spectrum of the fluctuating part of the energy level density. Instead of $K(\tau)$, we can use the power spectrum $P^n(\tau)$ of $\widetilde{n}(\epsilon)$ to analyze spectral fluctuations. It can be shown that for very large L values [5]

$$\langle P^n(\tau)\rangle = \frac{K(\tau)}{4\pi^2\tau^2}. \tag{5}$$

Then, we shall (i) sample $\widetilde{n}(\epsilon)$ for integer values of the energy and study the fluctuations of the new $\widetilde{n}_q$ function; (ii) map the independent variable of $\widetilde{n}_q$ (an energy) to the independent variable of δ_q (a dimensionless index).

Quantum Chaos.– One of the most important features of fully chaotic system is the universal behavior of their spectral fluctuations. Therefore, we can use RMT to obtain the main properties of $\langle P^n(\tau)\rangle$ in this kind of systems. Exact analytical expressions are known [6] for $K(\tau)$ in all RME, leading to

$$\langle P^n(\tau)\rangle_{\beta=2} = \begin{cases} \dfrac{1}{4\pi^2\tau}, & \tau \leq 1, \\ \dfrac{1}{4\pi^2\tau^2}, & \tau \geq 1, \end{cases} \tag{6}$$

$$\langle P^n(\tau)\rangle_{\beta=1} = \begin{cases} \dfrac{2-\log(1+2\tau)}{4\pi^2\tau}, & \tau \leq 1, \\ \dfrac{2-\tau\log\left(\dfrac{2\tau+1}{2\tau-1}\right)}{4\pi^2\tau^2}, & \tau \geq 1. \end{cases} \tag{7}$$

For small τ values, (6) and (7) become (see Ref. [7])

$$\langle P^n(\tau)\rangle_\beta = \frac{1}{2\beta\pi^2\tau}, \quad \tau \ll 1, \tag{8}$$

and for $\tau \geq 1$ we can approximate $\langle P^n(\tau)\rangle_\beta$ by

$$\langle P^n(\tau)\rangle_\beta \simeq \frac{1}{4\pi^2\tau^2}, \quad \tau \geq 1, \tag{9}$$

which is exactly equal for GUE and a good approximation for GOE.

On the other hand, using periodic orbit theory and semiclassical mechanics it is possible to calculate $K(\tau)$ for chaotic systems. The semiclassical expression essentially coincides with the results of RMT for $\tau_{min} \ll \tau \ll \tau_H$, where τ_{min} is the period of the shortest periodic orbit, and $\tau_H = h/\langle s\rangle$ is the Heisenberg time of the system, related to the time a wave packet takes to explore the complete phase space of the system [8, 9]. The expressions derived in the following are then directly applicable to generic chaotic quantum systems for τ between this two values.

The next step is to relate the spectral fluctuations of $\widetilde{n}_q$ to those of the continuous function $\widetilde{n}(\epsilon)$, as given by their power spectra. With the usual definitions for the continuous and discrete Fourier transforms we have

$$\widehat{n}_k = \frac{1}{\sqrt{N}} \sum_{q=-\infty}^{\infty} \widehat{n}\left(\frac{k}{N} + q\right). \tag{10}$$

Therefore, to relate P^n_k and $P^n(\tau)$ we need the precise knowledge of $\widehat{n}(\tau)$ for all τ. Since RMT only provides the mean value of its squared modulus, we introduce two simplifying assumptions: (a) for times $\tau = k/N$, $|\widehat{n}(k/N)| = \sqrt{N\langle P^n(k/N)\rangle}$; (b) the phases of $\widehat{n}(k/N)$ are random variables uniformly distributed in the interval $[0, 2\pi)$. These assumptions, specially (a), are reasonable as far as we are only interested in the average values of $P^n(\tau)$ and P^n_k. With these simplifications and using eq. (9) we obtain the following result for $N \gg 1$,

$$\langle P^n_k\rangle = \langle P^n(k/N)\rangle + \langle P^n(1-k/N)\rangle - \frac{N^2}{4\pi^2k^2} - \frac{N^2}{4\pi^2(N-k)^2} + \frac{1}{4\sin^2\left(\dfrac{\pi k}{N}\right)}, \quad k = 1, 2, \cdots, N-1. \tag{11}$$

Finally, from the relationship between the variances of δ_q and $\widetilde{n}(\epsilon)$ for chaotic systems [10] we obtain an analytical expression for $\left\langle P_k^\delta \right\rangle$ in terms of $\langle P^n(k/N)\rangle$; the final result is

$$\left\langle P_k^\delta \right\rangle = \langle P_k^n \rangle - \frac{1}{12}, \quad k = 1, 2, \cdots, N-1, \quad N \gg 1. \quad (12)$$

Eq. (12), together with (11) and (7) or (6), gives explicit expressions of $\left\langle P_k^\delta \right\rangle$ for GOE and GUE. For generic chaotic systems we can apply GOE or GUE results depending on their symmetries.

Integrable systems.– A similar calculation can be performed for integrable systems. In this case $K(\tau) = 1$ [11], and instead of (6) or (7) we have $P^n(\tau) = \frac{1}{4\pi^2\tau^2}$ [7] for all τ. Using (11), obtained with the same assumptions (a) and (b), and the fact that the variances of δ_q and $\widetilde{n}(\epsilon)$ are equal for integrable systems, we get

$$\left\langle P_k^\delta \right\rangle = \frac{1}{4\sin^2\left(\frac{\pi k}{N}\right)}, \quad N \gg 1. \quad (13)$$

In order to check the assumptions (a) and (b) introduced above, we may now give an alternative proof of eq. (13) without using these assumptions. For integrable systems, the spacings sequence s_i can be considered as a sequence of independent random variables with Poisson distribution [1]. Therefore, the δ_q statistic is a sum of such random variables $\delta_q = \sum_{i=1}^{q} w_i$, with $w_i = s_i - < s >$. Additionally, in this case we have $\langle \widehat{w}_k \rangle = 0$ and $\left\langle |\widehat{w}_k|^2 \right\rangle = 1$, where $\widehat{w}_k$ is the Fourier transform of w_i. Then, we can write

$$\delta_q = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \widehat{w}_k \sum_{m=1}^{q} e^{-\frac{2\pi i k m}{N}}. \quad (14)$$

This series can be added up considering that $\langle \widehat{w}_k \rangle = 0$; we obtain for $N \gg 1$

$$\delta_q = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \frac{i\widehat{w}_k e^{\frac{-i\pi k}{N}}}{2\sin\left(\frac{\pi k}{N}\right)} e^{-\frac{2\pi i k q}{N}}, \quad (15)$$

and therefore

$$\widehat{\delta}_k = \frac{i\widehat{w}_k e^{\frac{-i\pi k}{N}}}{2\sin\left(\frac{\pi k}{N}\right)}, \quad N \gg 1. \quad (16)$$

Consequently, taking into account that for a Poissonian sequence $\left\langle |\widehat{w}_k|^2 \right\rangle$=1, we can write

$$\left\langle P_k^\delta \right\rangle = \left\langle \left|\widehat{\delta}_k\right|^2 \right\rangle = \frac{1}{4\sin^2\left(\frac{\pi k}{N}\right)}, \quad N \gg 1. \quad (17)$$

The coincidence of eqs. (13) and (17) shows that the assumptions (a) and (b) introduced to derive (12) and (13) are sound approximations.

$1/f$ and $1/f^2$ noises.– Note that eqs. (12) and (13) are generic results, valid for every chaotic and integrable quantum system, respectively. Nevertheless, there can be some deviations at the lower frequencies due to the behavior of the shortest periodic orbits. When $k \ll N$ the first term of eq. (11) becomes dominant and using eq. (8) we can write, for chaotic systems,

$$\left\langle P_k^\delta \right\rangle = \frac{N}{2\beta\pi^2 k}, \quad k \ll N, \quad N \gg 1. \quad (18)$$

Similarly, for integrable systems, eq. (13) becomes

$$\left\langle P_k^\delta \right\rangle = \frac{N^2}{4\pi^2 k^2}, \quad k \ll N, \quad N \gg 1. \quad (19)$$

These expressions show that for small frequencies, the excitation energy fluctuations exhibit $1/f$ noise in chaotic systems and $1/f^2$ noise in integrable systems. As we shall see below, these power laws are also approximately valid through almost the whole frequency domain, due to partial cancellation of higher order terms in eqs. (12) and (13). Only near $k = N/2$, the so called Nyquist frequency [4], the effect of these terms becomes appreciable.

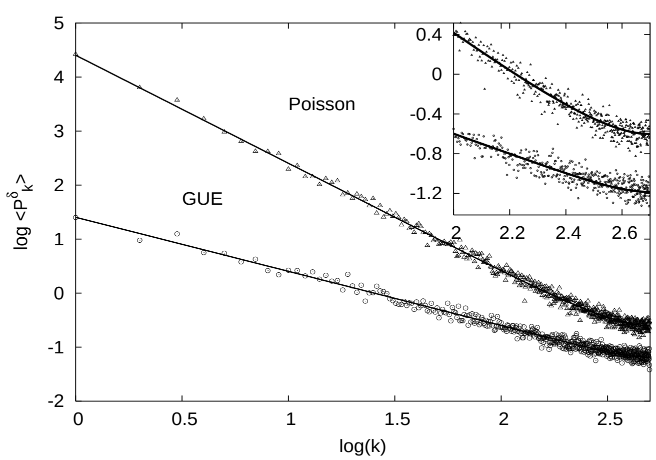

FIG. 1: Theoretical power spectrum of the δ_q function for GUE and integrable systems (solid lines), compared to the results calculated numerically using 30 GUE matrices of dimension $N = 10^3$ (circles) and 30 Poisson level sequences of length $N = 10^3$ (triangles).

To test all these theoretical expressions we have compared their predictions to numerical results obtained for different ensembles and systems. Fig. 1 displays the theoretical values of $\left\langle P_k^\delta \right\rangle$ for GUE and integrable systems, as given by (12) and (13), together with the numerical results for GUE matrices and Poisson level sequences published in [3]. In order to enlarge the high

frequency region, where the numerical results show small deviations from the $1/f^\alpha$ power law behavior, an upper right panel is added to the figure. The agreement between the theoretical and numerical results is excellent at all frequencies (note that there are no free parameters in the analytical result). The theoretical curve describes perfectly the power laws, characteristic of small and intermediate frequencies, as well as the deviations observed at the highest frequencies.

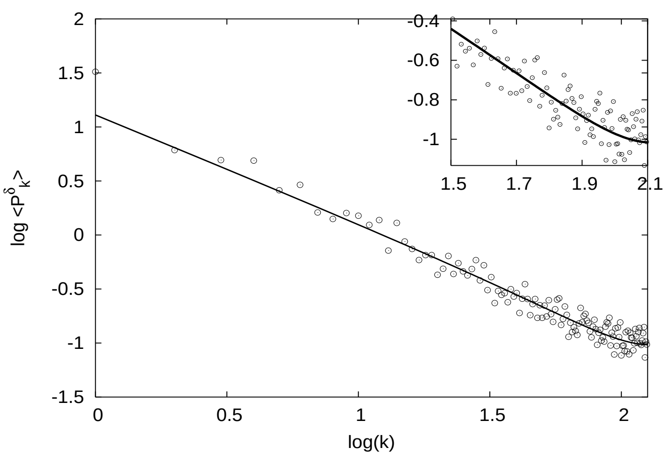

FIG. 2: Numerical average power spectrum of the δ_q function for ^{34}Na, calculated using 25 sets of 256 consecutive levels from the high level density region, compared to the parameter free theoretical values (solid line) for GOE.

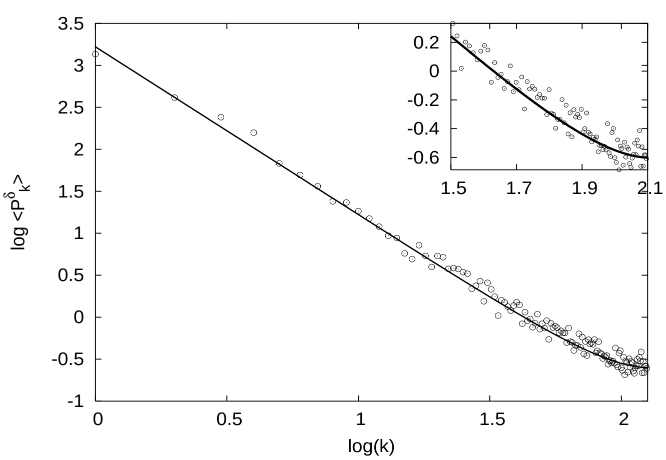

FIG. 3: Numerical average power spectrum of δ_q for a rectangular billiard, calculated using 25 sets of 256 consecutive levels, compared to the parameter free theoretical values (solid line) for integrable systems.

We have also compared our predictions to the power spectra of δ_q for two physical systems: an atomic nucleus (chaotic) and a rectangular billiard (integrable).

In the first case we have performed a shell model calculation for ^{34}Na using an adequate realistic interaction. The Hamiltonian matrices for different angular momenta, parity and isospin were fully diagonalized. Then, 25 sets of 256 consecutive high energy levels of the same $J^\pi T$ were selected, and the average power spectrum of the δ_q function was calculated. Fig. 2 shows the result of this calculation together with the theoretical values of eq. (12). An excellent agreement between the theoretical and numerical results is obtained through the whole frequency interval.

As an example of integrable system we have chosen a rectangular billiard with sides of length $a = \sqrt{\lambda}$ and $b = 1/\sqrt{\lambda}$, with $\lambda = (\sqrt{5}+1)/2$; this geometry gives rise to an irrational ratio $a/b = \lambda$, and thus there are no degeneracies in the spectrum. We have calculated the spectrum and selected 25 sets of 256 consecutive very high energy levels in order to avoid, as far as possible, the influence of short periodic orbits. Fig. 3 shows the results for the average power spectrum of δ_q and the theoretical values given by eq. (13). As in previous cases, we can see that the agreement between the theoretical and numerical results is very good in the whole frequency domain.

In summary, we have derived theoretical expressions for the power spectrum of the δ_q function both for regular and chaotic quantum systems. These expressions are universal and do not contain any free parameter. We have compared our theoretical predictions with numerical results for RME, a rectangular billiard and an atomic nucleus, obtaining an excellent agreement for all these systems. The theory reproduces the power laws of type $1/f$ for chaotic systems, and $1/f^2$ for regular ones, observed in the power spectrum of the excitation energy fluctuations up to frequencies very close to the Nyquist limit.

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